A Numerical study of the Forced Damped Pendulum

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Abstract: Several cases of the *Forced damped pendulum* are numerically demonstrated using the mathematical modeling package *MLAB*. A Poincaré return map for the chaotic case is also given.

An ideal pendulum (*i.e.* with no friction) will swing back and forth (or loop in a full circle) forever if there is no outside force other than gravity acting upon it. Moreover, a pendulum with friction will come to rest if there is no other outside force besides gravity acting upon it. A more general forced damped pendulum with a periodic driving force pushing it shows more interesting asymptotic behavior than these two trivial cases. The angular position in radians as a function of time $\theta(t)$ of a forced damped pendulum is described by the following second order differential equation.

$$\frac{d^2\theta}{dt^2} + \nu\frac{d\theta}{dt} + sin\theta = \rho sin(2\pi ft), \ \ \theta(0) = \theta_0, \ \ \frac{d\theta}{dt}(0) = s, \qquad (1)$$

where $\frac{d^2\theta}{dt^2}$ represents the inertia, $\nu\frac{d\theta}{dt}$ represents friction at the pivot, $sin(\theta)$ represents gravity, and $\rho sin(2\pi f t)$ represents a sinusoidal frequency f driving torque applied at the pivot. θ_0 is the initial angular position and s is the initial angular velocity of the pendulum.

Numerical solutions show that both chaotic and periodic solutions of the forced damped pendulum equation are possible depending on the particular choice of system parameters ν, ρ and f.

Torque

θ

M

Forced, damped pendulum.

Gravity
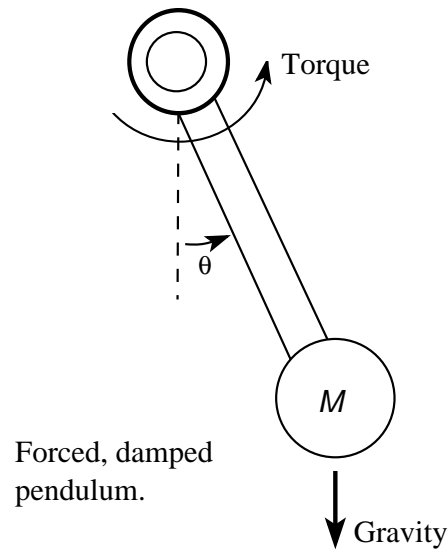
If we want to see how the pendulum is actually moving, we can solve the differential equation and plot the angle variable θ against time t. We will use the facilities of the mathematical modeling system *MLAB* to solve this differential equation and display the solution. The following is an *MLAB* dofile(script) that solves the forced damped pendulum equation.

```
"pendulum.do — Solve the ODE for a forced pendulum"

"the differential equation for the forced damped pendulum"
fct theta"t(t) = rho*sin(t) - c1*theta't - sin(theta)
"rho*sin(t) is the driving force, c1*theta't is friction, sin(theta) is gravity"

"Read-in parameter values"
type "input the friction coefficient c1"; c1 = kread();
type "input the driven force amplitude" rho = kread();
type "input n: number of units of time to solve for"; n = kread();

"Read-in the initial conditions"
type "Specify the initial conditions"
type "input theta(0)"; theta0 = kread(); initial theta(0) = theta0
type "input theta'(0)"; thetap0 = kread(); initial theta't(0) = thetap0

"Specify the time values where we want the solution"
```

data = 0:n:.1; "get an array from 0 to n at every .05"

/* Now, integrate the differential equation. This is done implicitly
inside the operator 'points', the result matrix is put in the array m.
The first column of m is the data array generated above, the second
column is the theta value at the time corresponding to the time value in
the first column, the third column is the theta't value at the time
corresponding to the time value in the first column. */
m = points(theta, theta't,data)

"draw the trajectory (angular position vs. time)"
draw m col 1;2, color red
top title "Regular"
left title "Cumulative Angular Position";
bottom title "Time"
view; "view the picture"

The following pictures are generated by variations of this *MLAB* dofile. They
are plots of time against the angular position. The left picture is the trivial
regular case with a periodic orbit where $c1 = 0$ and $rho = 0$. The right
picture shows a chaotic orbit where $c1 = 0.3$ and $rho = 2.5$. In all cases,
the initial point corresponds to the initial conditions denoted by *theta0* and
theta1 in the dofile.
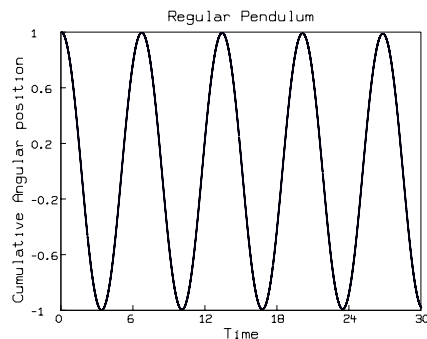
*Pendulum with no friction, no exterior*
force. c1 = 0, rho = 0. initial point
(1.0,0)
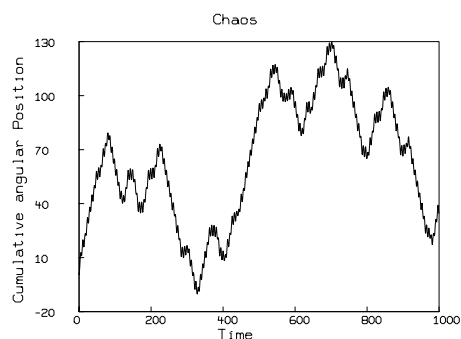
*Chaotic orbit, no regular pattern observed,*
c1 = 0.3, rho = 2.5. initial point (0.5,
2.0)

Here are another pair of pictures that are generated by the same dofile. The
left picture shows a transient chaotic behavior which settles down to a peri-
odic orbit after awhile. The parameter values are $c1 = 0.1$, $rho = 1.5$. The

3

right picture shows another kind of behavior. It is not periodic, since the angle keeps increasing, it is not chaotic either since the the angle is increasing in a regular pattern, thus the Lyapunov exponent is not positive. The parameter values are $c1 = 0.2$, $rho = 2.5$.
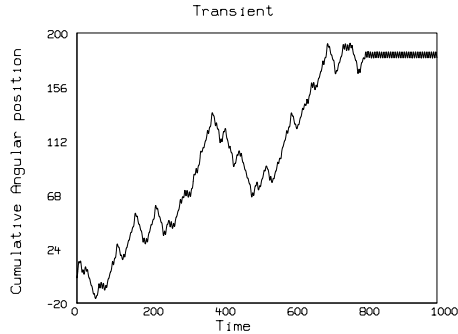

Transient, started chaotic and settles down to periodic. c1 = 0.1, rho = 1.5, initial point (0.5, 2.0)
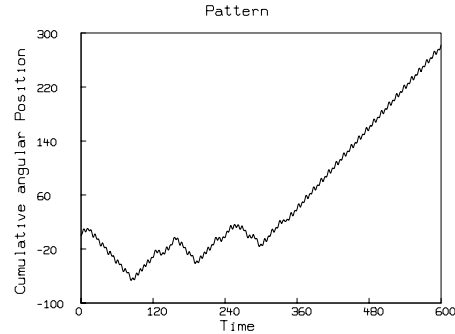
Not periodic, not chaotic, angle keeps increasing. c1 = 0.2, rho = 2.5, initial point (0.5, 0.9)

To see a trajectory of a point in the phase-space $(t, \theta(t), \theta'(t))$ for the forced damped pendulum, we will construct an object called the *Poincaré* map which is often used to reduce a continuous time system (or "flow") to a discrete time map with one less dimension. The basic idea of a *Poincaré* map is to choose some appropriate hypersurface in the phase space and observe the intersection of the orbit in the phase-space with the surface. Since the solution of the system is unique with a given initial point, when we neglect numerical round-off error, each intersection point will uniquely determine the successive point. Thus, a continuous flow is reduced to a discrete map with one less dimension.

4
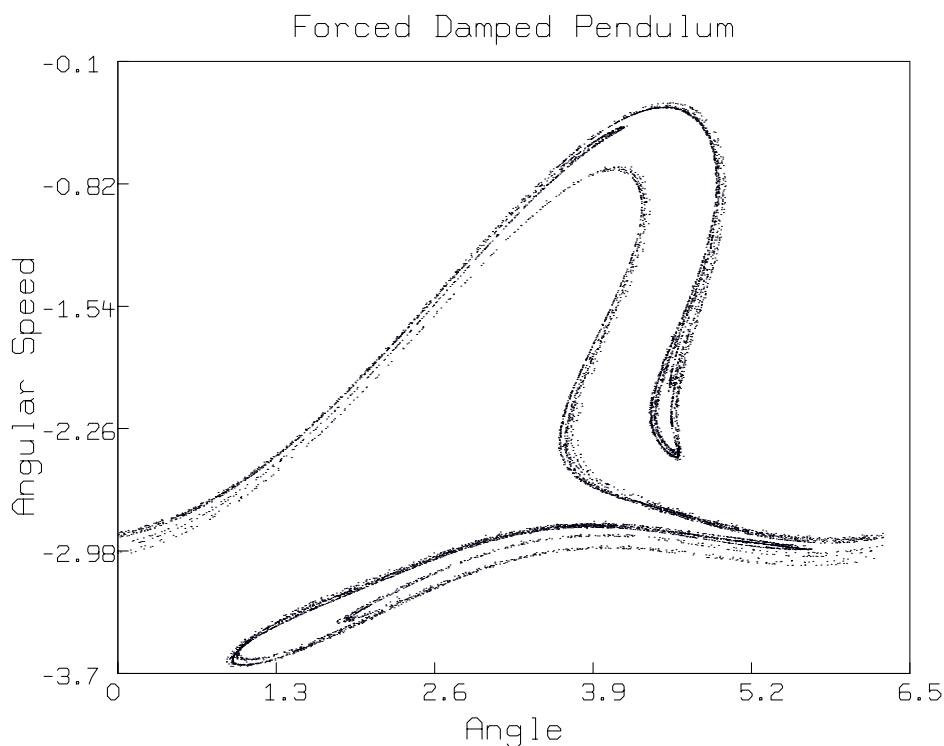
Poincaré return map for the pendulum, the section plane is
t = 0, c1 = 0.3, rho = 2.5. n = 10000. initial point (0.5, 2.0)

The picture above is the *Poincaré* map of the trajectory corresponding to the initial point $(0.5, 2.0)$ in the $(\theta(t), \theta'(t))$ reduced-dimension phase space for our forced damped pendulum system with the parameter $rho = 2.5$ and $c1 = 0.3$ which is the same as the chaos picture above. We used $n = 10000$. It is the *Poincaré* return map with the section surface $t = 0$, which is equivalent to $t = 2k\pi$ for any integer k since the variable t only appears in $sin(t)$, and for any integer value of k, $sin(2k\pi)$ will have the same value 0, *i.e.* our reduced-dimension phase space is a cylinder, which is displayed in an "unwrapped" form above.

To compute the above *Poincaré* map, we used almost the same *MLAB* dofile with only a few minor changes. We only want the solution at every time when $t = 2k\pi$ (a multiple of 2π), Thus, we need to substitute the line for computing the time-list data in the above *MLAB* dofile with the following line:

data = 0:(pi2*n):pi2; "get an array from 0 to pi2*n at every pi2"

Also, we need to plot different columns of the solution matrix m, thus, we must substitute the draw statement lines with the following lines

```
/* The variable theta is the angle of the arm of the pendulum. The angle
can range from 0 to pi2, any angle that is outside this region has a
corresponding angle in this region. We define a function to map all
the angles into 0:pi2 */
fct ft(theta) = mod(theta, pi2)
m col 2 = ft on (m col 2); "map col 2 into the region 0 to pi2"
"draw a dot at each phase plane at each point (theta,theta't)"
draw m col 2:3, color red, lt none, pt dotpt
top title "Forced Damped Pendulum"
left title "Angular Velocity"; bottom title "Angle"
view; "view the picture"
```

One can easily generate further pictures with other parameter values. More information about *MLAB* is available from Civilized Software Inc. at the address given above.