Filtering Signals

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Fourier Series and Transforms for Non-discrete Periodic Functions

J. B. Fourier proposed that, under mild conditions, a real-valued periodic period-p function, $x(t) = x(t+p)$, can be expressed as the sum of sinusoidal oscillations of various frequencies, amplitudes, and phaseshifts, so that

$$x(t) = \sum_{h=0}^{\infty} M(h/p) \cdot \cos\left(2\pi(h/p)t + \phi(h/p)\right).$$

This series is called the real Fourier series of the period-p function x. The term $M(h/p) \cdot \cos(2\pi(h/p)t + \phi(h/p))$ is a cosine oscillation of period p/h, frequency h/p, amplitude $M(h/p)$, and phaseshift $\phi(h/p)$. The function x determines and is determined by the amplitude function M and the phase function ϕ, which are both defined on the frequency values $\{0, 1/p, 2/p, \ldots\}$.

It is convenient to use Euler's relation $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ to develop the mathematical theory of Fourier series for complex-valued functions, rather than just real-valued functions. In this case, we can express x in terms of an associated discrete complex-valued function $x^\wedge$ which contains the amplitude and phase functions combined together. The complex-valued function $x^\wedge$ is defined on the discrete set $\{\ldots, -2/p, -1/p, 0, 1/p, 2/p, \ldots\}$. This function $x^\wedge$ is called the Fourier transform of x.

In particular, the *Fourier transform* of the periodic period-p function x is:

$$x^\wedge(s) := (1/p) \int_{-p/2}^{p/2} x(t) e^{-2\pi i s t}\, dt,$$

for $s = \ldots, -2/p, -1/p, 0, 1/p, 2/p, \ldots$.

The *inverse Fourier transform* of $x^\wedge$ is:

$$x^{\wedge\vee}(t) := \sum_{h=-\infty}^{\infty} x^\wedge(h/p) e^{2\pi i (h/p) t} = x(t) \quad \text{a.e.}$$

This sum is the Fourier series of x; it is a sum of terms made-up of the so-called *Fourier coefficients* $x^\wedge(h/p)$ times the complex oscillations $e^{2\pi i(h/p)t}$. The Fourier transform of x thus produces the Fourier coefficients of x.

Fourier Series and Transforms for Discrete Periodic Functions

When we have sampled a function at the times $0, T, 2T, \ldots, (n-1)T$, for a total of n samples equally-spaced with step-size T, we may treat these samples as a discrete function x, and we may extend this function periodically so that the discrete Fourier transform discussed below may be employed.

Thus let $x(t)$ be a discrete complex-valued periodic function of period p defined at $t = \ldots, -2T, -T, 0, T, 2T, \ldots$, with $p = nT$. Thus either of the discrete ranges $0 \le t \le (n-1)T$, or $-\lfloor n/2 \rfloor T \le t \le (\lceil n/2 \rceil - 1)T$, among others, constitutes one period. Of course, x may in fact be defined on the whole real line.

The discrete Fourier transform of the discrete period-p, step-size T function x is

$$x^\wedge(s) = (T/p) \sum_{h=-\lfloor n/2 \rfloor}^{\lceil n/2 \rceil - 1} x(hT) e^{-2\pi i s h T},$$

for $s = \ldots, -2/p, -1/p, 0, 1/p, 2/p, \ldots$, and $x^\wedge(s)$ is defined to be 0 for s not an integral multiple of $1/p$. The transform $x^\wedge$ is a discrete periodic function of period n/p with step-size $1/p$. The ratio of the period and the step-size is the same value n for both x and $x^\wedge$.

This sum is just a rectangular Riemann sum approximation of the integral form of the Fourier transform of an integrable function x, defined on a regular mesh of points, each of which is T units apart from the next.

Unlike the transform of a non-discrete periodic function defined on the entire real line, $x^\wedge$ is also a (discrete) *periodic* function, and hence the inverse operator, $\vee$, acts on the same type of functions as the direct operator, $\wedge$, but, in general, with a different period and step-size. When necessary we shall write $\wedge(p;n)$ and $\vee(p;n)$ to denote the discrete Fourier transform and the inverse discrete Fourier transform for discrete periodic functions of period p with step-size p/n.

The inverse discrete Fourier transform of x of period p with step-size $T = p/n$ is

$$x^{\vee(p;n)}(r) = \sum_{h=-\lfloor n/2 \rfloor}^{\lceil n/2 \rceil - 1} x(hT) e^{2\pi i r h T},$$

for $r = \ldots, -2/p, -1/p, 0, 1/p, 2/p, \ldots$.

By convention, $x^{\wedge(p;n)}(s) = 0$ and $x^{\vee(p;n)}(s) = 0$ except possibly at $\ldots$, $-2/p$, $-1/p$, 0, $1/p$, $2/p$, $\ldots$. Note that $\vee(n/p;n)$ is the inverse operator of $\wedge(p;n)$, and $\vee(p;n)$ is the inverse operator of $\wedge(n/p;n)$. When $\wedge$ is understood to be $\wedge(p;n)$, $\vee$ shall normally be understood to be $\vee(n/p;n)$. For the particular case where x is of period $p = nT$ with step-size p/n, both $x^{\wedge(p;n)}$ and $x^{\vee(p;n)}$ are periodic of period $n/p = 1/T$, and are defined on a mesh of step-size $1/p$.

For x of period $p = nT$, the inverse discrete Fourier transform of $x^{\wedge}$ is

$$x^{\wedge\vee}(t) = \sum_{h=-\lfloor n/2\rfloor}^{\lceil n/2\rceil-1} x^{\wedge}(h/p)e^{2\pi i(h/p)t},$$

where $x^{\wedge\vee}$ is of period p defined on a mesh of step-size $T = p/n$, and $x^{\wedge\vee}(t) = x(t)$ for $t = \ldots, -2T, -T, 0, T, 2T, \ldots$. This is the Fourier series of the discrete function x.

For $-\lfloor n/2\rfloor \leq h \leq \lceil n/2\rceil - 1$, $x^{\wedge}(h/p)$ is the complex amplitude of the complex oscillation $e^{2\pi i(h/p)t}$ of frequency h/p cycles per t-unit in the Fourier series $x^{\wedge\vee}$, and $x^{\wedge\vee}$ is a sum of complex oscillations of frequencies $-\lfloor n/2\rfloor/p, \ldots, 0, \ldots, (\lceil n/2\rceil - 1)/p$. Thus $x^{\wedge\vee}$ is *band-limited*; that is the Fourier series $x^{\wedge\vee}$ has no terms for frequencies outside the finite interval or band $[-\lfloor n/2\rfloor/p, (\lceil n/2\rceil - 1)/p]$.

Note $x^{\wedge\vee}(t)$ is defined for all t; it is a periodic function of period p which coincides with x at $t = \ldots, -2T, -T, 0, T, 2T, \ldots$. Indeed the function $x^{\wedge\vee}$ is the unique period-p periodic function in $L^2(Q)$ with this property which is band-limited with $x^{\wedge(p;n)\vee(n/p;n)\wedge(p;n)}(s) = 0$ for s outside the band $[-\lfloor n/2\rfloor/p, (\lceil n/2\rceil - 1)/p]$. If x is real, then when n is odd, $x^{\wedge\vee}$ is real, but when n is even, $x^{\wedge\vee}$ is complex in general, even though $x^{\wedge\vee}(t)$ is real when t is a multiple of T.

Another useful form of the discrete Fourier Inversion theorem is

$$x^{\wedge\vee}(kT) = x(kT) = \sum_{h=-\lfloor n/2\rfloor}^{\lceil n/2\rceil-1} x^{\wedge}(h/p)e^{2\pi i(h/n)k}.$$

For $nT = p$, the functions $x(hT)$ and $e^{-2\pi i shT}$ with s a multiple of $1/p$ are both periodic functions of h with period n, and hence the discrete Fourier transform of x can be obtained by summing over any contiguous index set of length n, so that $x^{\wedge}(s) = (T/p)\sum_{h=a}^{n-1+a} x(hT)e^{-2\pi i shT}$ for $s = \ldots, -1/p, 0, 1/p, \ldots$. Similarly, $x^{\wedge}(h/p)$ and $e^{2\pi i(h/p)t}$ with t a multiple of T are both periodic functions of h with period n, so $x^{\wedge\vee}(t) = \sum_{h=a}^{n-1+a} x^{\wedge}(h/p)e^{2\pi i(h/p)t}$, for $t = \ldots, -2T, -T, 0, T, 2T, \ldots$.

Indeed, the periodicity insures the *same* values are being summed, regardless of the value of a, so the Fourier series denoted by $x^{\wedge\vee}$ is a unique sum of complex oscillations, which, when expressed in the particular form where $a = -\lceil n/2 \rceil$, allows us, in the case where x is real, to easily combine the positive and negative frequency terms (taking $x^\wedge(n/(2p)) = 0$ when n is even) and shows the spectral decomposition of $x^{\wedge\vee}$ to be

$$x(t) = \sum_{h=0}^{\lfloor n/2 \rfloor} M(h/p)\cos(2\pi(h/p)t + \phi(h/p)).$$

The functions $M(n/(2p))$ and $\phi(n/(2p))$ are defined as follows.

$$M(h/p) = \sqrt{(x^\wedge(h/p) + x^\wedge(-h/p))^2 - (x^\wedge(h/p) - x^\wedge(-h/p))^2}/(1 + \delta_{h0})$$

for $0 \le h \le \lceil n/2 \rceil - 1$ and, when n is even, $M(n/(2p)) = |x^\wedge(-n/(2p))|$, and, by definition, $M(h/p) = 0$ for $h > \lfloor n/2 \rfloor$. Also

$$\phi(h/p) = \text{atan2}(-i(x^\wedge(h/p) - x^\wedge(-h/p)), x^\wedge(h/p) + x^\wedge(-h/p)).$$

for $0 \le h \le \lceil n/2 \rceil - 1$, and, when n is even, $\phi(n/(2p)) = \text{atan2}(0, x^\wedge(-n/(2p)))$, and $\phi(h/p) = 0$ for $h > \lfloor n/2 \rfloor$.

The discrete positive function $M(h/p)$ is the *amplitude spectrum function* of x, and its square, $M(h/p)^2$ is the *power density spectrum function* of x. Since $M(h/p)$ is the amplitude of the oscillation-component of x of frequency h/p, both of these functions show the relative "amounts" of each oscillation-component contained in x.

Filtering

The process of *filtering* the discrete function x consists of modifying the amplitudes of its oscillatory components, so that $M(h/p)$ is changed to $z(h/p)M(h/p)$ for some desired scale-factor $z(h/p)$. Note when $z(h/p) = 0$, the oscillation-component of frequency h/p is entirely eliminated. If we reduce or eliminate the oscillation-components of x above a given "cut-off" frequency, we have applied a *low-pass* filter which "passes" the low-frequency components and "stops" the higher-frequency components. A *high-pass* filter does just the opposite, while a *band-pass* filter passes those components whose frequencies lie in a specified interval.

It is convenient to perform filtering in the complex domain with respect to the Fourier series of x. This is because the convolution theorem $(x * y)^\wedge = x^\wedge y^\wedge$ (* denotes the convolution operation) can then be employed. The basic idea in this case is as follows.

1. Given the n complex sample values $x(0), x(T), \ldots, x((n-1)T)$, sampled with the step-size T, compute the n complex Fourier coefficient values

x^$(-\lfloor n/2 \rfloor/p)$, ..., x^$(-1/p)$, x^(0), x^$(1/p)$, ..., x^$((\lceil n/2 \rceil - 1)/p)$, where $p = nT$.

2. Choose the filter-coefficient values $z(-\lfloor n/2 \rfloor/p), \ldots, z(-1/p), z(0), z(1/p), \ldots, z((\lceil n/2 \rceil - 1)/p)$.

3. Compute the "filtered" Fourier coefficients $f(j/p) := z(j/p)x$^(j/p) for $-\lfloor n/2 \rfloor \le j \le \lceil n/2 \rceil - 1$.

4. Compute the inverse discrete Fourier transform $f^\vee$ of the discrete function f computed in step 3. The result is the filtered form of the discrete function x.

Generally the filter-coefficients should be chosen to be symmetric about 0; otherwise the phase-shifts of the oscillation-components of x will be modified by the act of filtering. Note that the function $z^\vee$ is the *impulse-response* function of a linear system that does the filtering specified by z via convolution. For a low-pass filter, we want $z(j/p)$ to be small or zero for $| j | > c$ for some cut-off index c.

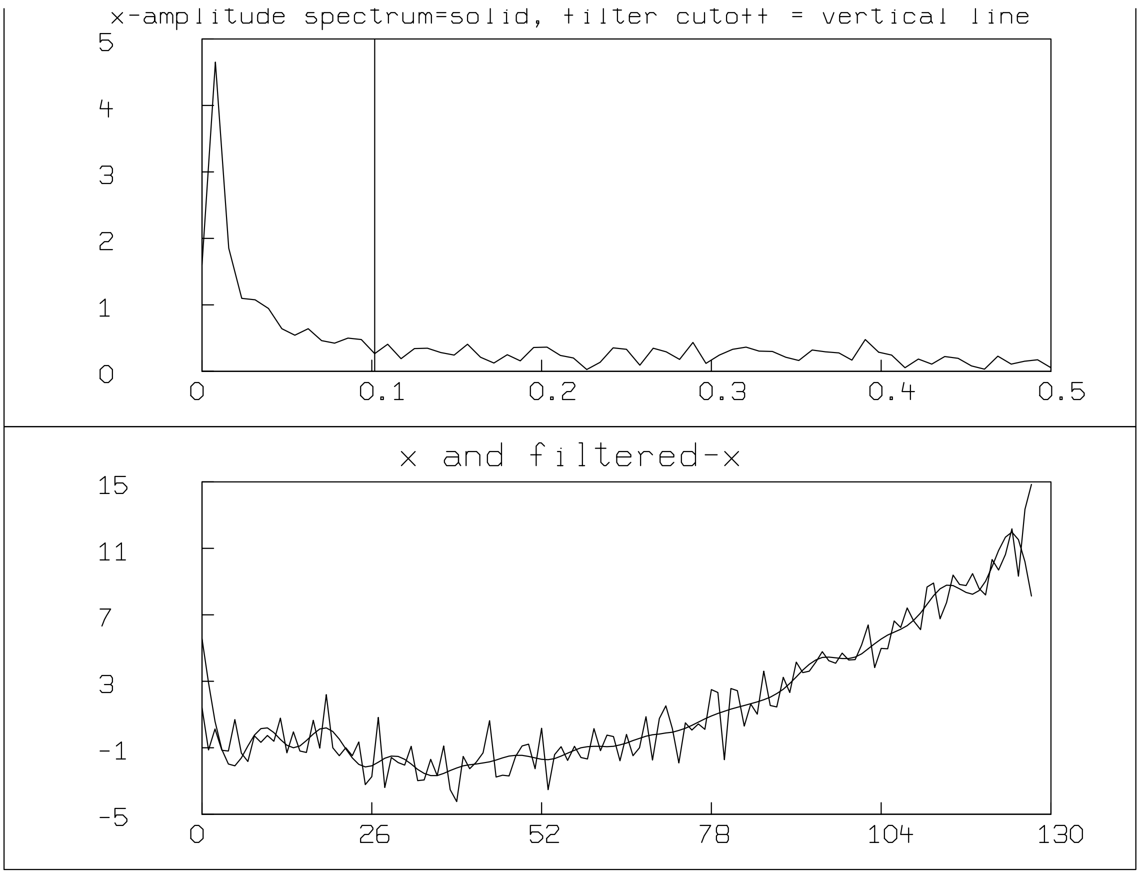
In MLAB, the complex Fourier transform and inverse transform operators *interpolate* their inputs so that, unless n is a power-of-two, n is decreased to the next-lower power-of-two less than n. The number of samples are thus decreased, when necessary, to obtain a power-of-two number of samples. This is needed in order to be able to apply the power-of-two fast Fourier transform. Beware; such interpolation can be harmful unless the original signal was sampled sufficiently frequently to avoid aliasing problems even after being thinned by interpolation. It is safest to simply sample a power-of-two number of points initially.

Here is an example in MLAB of applying a pure low-pass filter to a discrete real signal x composed of n samples taken with step-size T. We will discard all but the k lowest-frequency components of the the $\lceil n/2 \rceil$ oscillation components contained in x. The real signal x is assumed to be given as a 2-column matrix `x`, where $\mathtt{x}[j, 1] = jT$ and $\mathtt{x}[j, 2] = x(jT)$.

Now the do-file given below generates an example noisy signal in the matrix `x` consisting of 128 samples with step-size 1, and then the signal in the matrix `x` is filtered to keep only the 14 lowest-frequency oscillation components.

```
/* do-file: filter.do = example of pure low-pass filtering */
reset
echodo = 3

/generate noisy signal */
fct f(t) = (t/60)^3+3*(t/60)^2-5*(t/60)+normran(0)
```

```
x = points(f,0:127)

k = 14 /* keep only the 14 lowest-frequency components */

tx=dft(x&'0)  /* A 0-column is attached for the imaginary part. */

n = nrows(x)
n1 = 2^floor(log2(n))

/* construct z = sequence of filter coefficients */
z[n1]=0
z row (n1/2-k+2):(n1/2+k)=1
z=z&'0

/* Do the filtering and construct the filtered output fx */
tx col 2:3 = cprod(tx col 2:3, z)
fx = idft(tx) col 1:2

/* Draw graphs comparing fx to x, Also draw the x amplitude-spectrum */
draw x
draw fx color green
top title "x and filtered-x"
frame 0 to 1, 0 to .5
w1=w

draw realdft(x) col 1:2
a = (13/128)&'0  /* 13/128 = the filter cut-off frequency for x */
draw a pt uband color green
top title "x-amplitude spectrum=solid, filter cutoff=vertical line" size .015
frame 0 to 1, .5 to 1
view
/* end of filter.do */
```



Note the filtered signal deviates from the original signal near the start and end. This is because the periodic extension of our signal x has a large difference between the ending sample-value and the follow-on starting sample-value so that there is a "discontinuity" in the periodic extension of x. Such discontinuities are fit in the Fourier series of x by high-frequency components, which, when removed, show the underlying tendency by the low-frequency components to smooth away the discontinuity. The behavior of the Fourier series of x in the neighborhood of a discontinuity is called Gibb's phenomenon.

In order to avoid spurious high-frequency oscillation-components in the signal due to Gibb's phenomenon which arise when our signal has a large difference between the starting sample-value and the ending sample-value, if our signal is ergogic, we could insure that x is an *even* function (*i.e.*, $x(t) = x(-t)$) which has no periodic-extension discontinuities by constructing a matrix twice the size (less one) of the originally-provided data matrix `x` as follows.

```
n = nrows(x)
x = (x row n:2)&x
k = 2*k
```

Often we will want to avoid using a pure low-pass filter as shown above, and, instead, taper the filter coefficients from 1 to 0 over a small transition region. There are a large variety of filter coefficient functions z and corresponding filter impulse-response functions $z^{\vee}$ that can be employed.

Rather than use Fourier transforms to low-pass filter a signal via a sequence of filter coefficients, we may, instead, directly *smooth* the signal via a variety of methods. Such smoothing is, of course, equivalent to using associated filter-coefficient functions, but the filtering is done directly by "convolution" in the time-domain, rather than multiplication in the frequency-domain. In MLAB, the operators SMOOTH, MMEAN, and SMOOTHSPLINE are all suitable methods to directly filter a signal. These operators should generally be compared with other filtering approaches when processing experimental data.