File: /General/MLAB-Text/Papers/lrc/lrc.tex

Electric Circuit Dynamics

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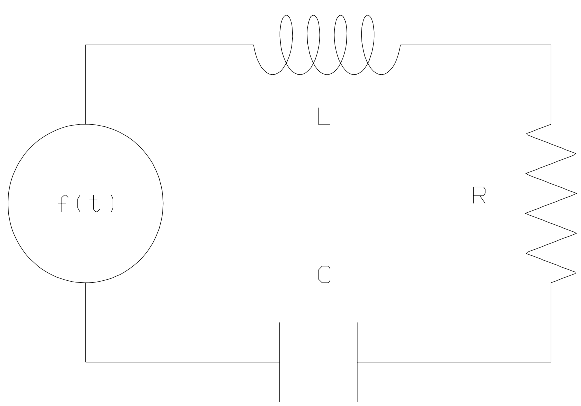
This example demonstrates MLAB's differential equation-solving facilities, the use of MLAB's Fourier transform operations, and MLAB graphics in the context of the classic problem of analyzing an LRC circuit. A circuit containing a coil with an inductance of L henries, a resistor with a resistance of R ohms, and a capacitor with a capacitance of C farads in series is traditionally called an LRC circuit. Consider the LRC circuit shown below which also contains a voltage source component which exhibits a voltage drop of $-f(t)$ at time t across the voltage source component. (This picture was constructed using MLAB.)



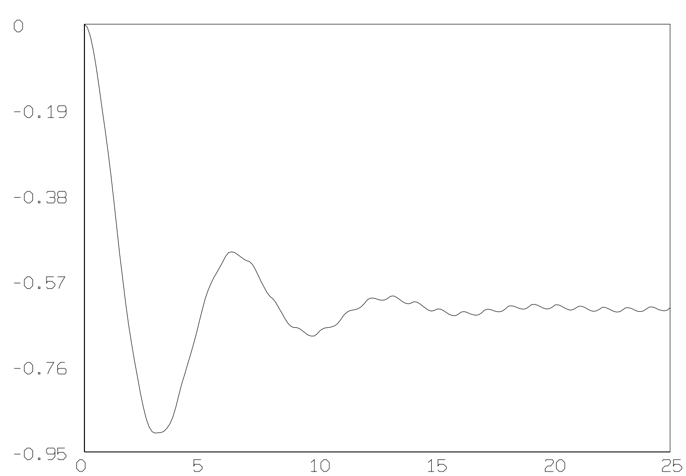
When the switch is closed at time 0, a current will begin to flow. Let $I(t)$ be the current flowing in the circuit at time t, measured in amperes. Let $Q(t)$ be the charge on the capacitor at time t, measured in coulombs. We have $dQ(t)/dt = I(t)$.

The voltage drop across the resistor at time t is given by Ohm's law as $R \cdot I(t)$. The voltage drop across the capacitor at time t is $Q(t)/C$. The voltage drop across the coil at time t is $L(dI(t)/dt)$, and by Kirchhoff's first law, the sum of the voltage drops across each of the circuit components is 0. Thus, we have

$$\begin{aligned} L\frac{dI(t)}{dt} + \frac{Q}{C} + RI(t) - f(t) &= 0, \qquad \text{or} \\ L\frac{d^2I(t)}{dt^2} + R\frac{dI(t)}{dt} + \frac{I(t)}{C} - \frac{df(t)}{dt} &= 0. \end{aligned}$$

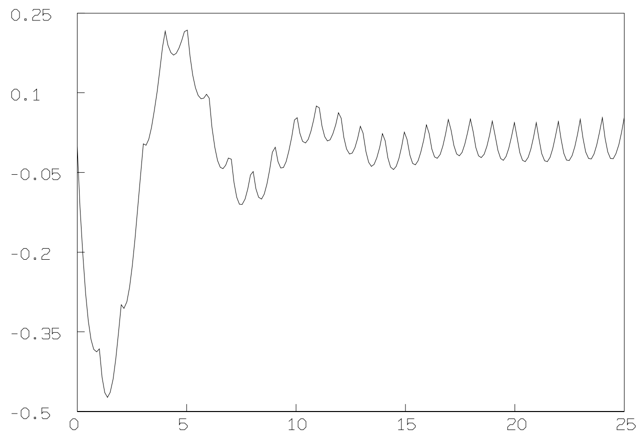
Take the initial conditions to be $I(0) = 0$ and $dI(0)/dt = 0$, and define $f(t) = \exp(-(t \bmod 1))$. Fix $L = 1$, $C = 1$, and $R = .5$. Now we may solve the differential equation defining the current flow function $I(t)$ and produce a graph of this function as shown below.

```
* function i''t(t)=(f't(t)-r*i't -i/c)/l
* initial i(0) = 0
* initial i't(0) = 0
* function f(t)=exp(-mod(t,1))
* l=1; c=1; r=.5
* m = integrate(i't,i''t,0:25!200)
* type odestr
   odestr = t i't i't't i i't
* draw m col (1,4)
* view
```



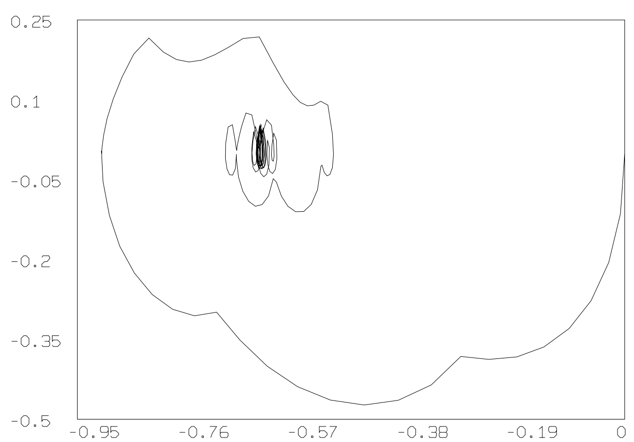
The value of the string variable ODESTR tells us what functions are numerically tabulated in the successive columns of M. The graph of the rate-of-change of the current function $I't(t)$ can also be plotted from the data in M.

```
* delete w
* draw m col 1:2
* view
```



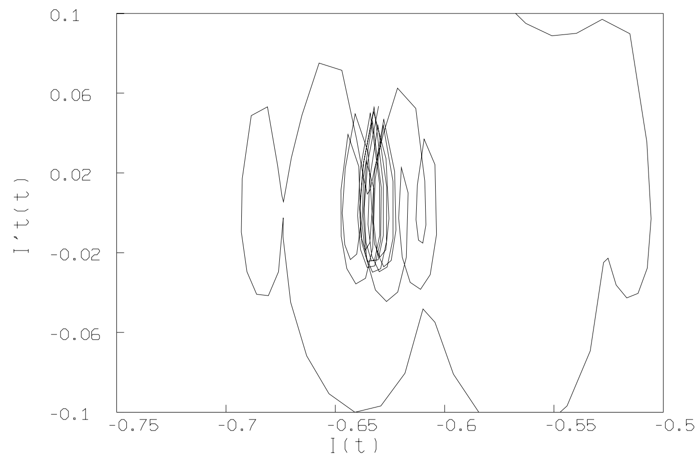
We can display the phase diagram graph for $I(t)$ by plotting $I't(t)$ vs. $I(t)$ as follows

```
* delete w
* draw m col (4,2)
* view
```



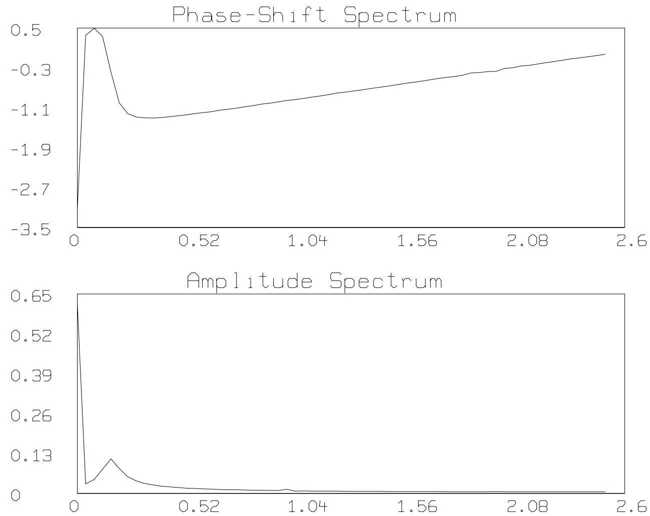
We may “zoom-in” to see the neighborhood of the limit cycle more clearly as follows.

```
* WINDOW -.75 TO -.5, -.1 TO .1
* VIEW
```



We can use the particular MLAB Fourier transform operator **realdft** to compute the amplitude and phase-shift spectra of the current function $I(t)$ tabulated in `M COL (1,4)`.

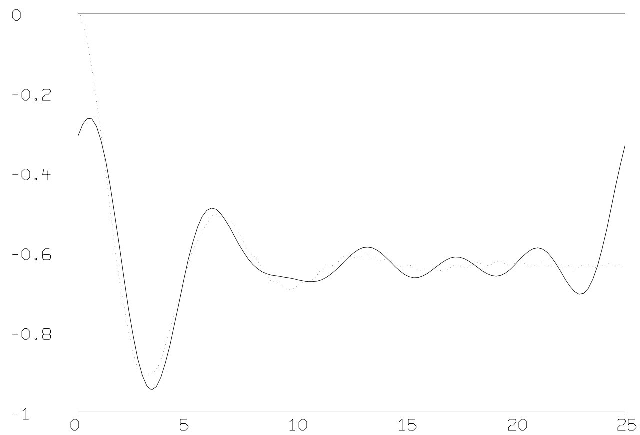
```
* d=realdft(m col(1,4))
* delete w
* draw d col 1:2
* frame 0 to 1, 0 to .5
* top title "Amplitude Spectrum" size .2 inches
* w1=w

* draw d col(1,3)
* frame 0 to 1, .5 to 1
* top title "Phase-Shift Spectrum" size .2 inches
* view
```



Ignoring the large DC value at frequency zero, we see that the maximum amplitude occurs at about .2 Hertz; this is the resonant frequency of the circuit. To see the amplitude spectrum at higher "resolution", we should subtract the DC-value from our signal $I(t)$ and compute the amplitude spectrum of this shifted signal whose mean value is now zero.

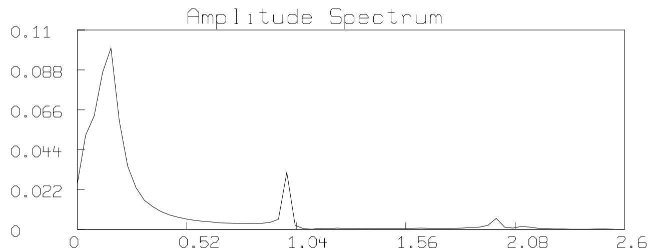
The Fourier transform of $I(t)$ contains the information to construct $I(t)$ as a periodic function via its Fourier series. If the Fourier series is truncated, the resulting sum is a filtered form of $I(t)$ omitting the high-frequency components corresponding to the truncated terms. Below we show a graph of the Fourier series of $I(t)$ truncated to 7 terms. Note that Gibbs' phenomenon is exhibited, showing non-uniform convergence to the mid-point of the discontinuities occuring at the points between successive periods of $I(t)$.

```
* fct s(t)=sum(i,1,n, d(i,2)*cos(2*pi*d(i,1)*t + d(i,3)) )
* n=7
* q=points(s,0:25!120)
* delete w,w1
* draw m col 1:2 lt dotted
* draw q
* view
```



We can also use the MLAB Fourier transform operator to compute the amplitude and phase-shift spectra of the current rate-of-change function $I't(t)$ tabulated in `M COL (1,2)`. As above, this amplitude spectrum shows the resonant frequency of the circuit to be about .2 Hertz. The peak at 1 Hertz corresponds to the frequency of oscillation of the forcing function $f(t)$.

```
* d=realdft(m col 1:2)
* delete w,w1
* draw d col 1:2
* frame 0 to 1, 0 to .5
* top title "Amplitude Spectrum" size .2 inches
* w1=w

* draw d col(1,3)
* frame 0 to 1, .5 to 1
* top title "Phase-Shift Spectrum" size .2 inches
* view
```



Just as before, the Fourier transform of $I't(t)$ contains the information to construct $I't(t)$ via its Fourier series. If the Fourier series is truncated, the resulting sum is a filtered form of $I't(t)$ omitting the high-frequency components corresponding to the truncated terms. Below we show a graph of the Fourier series of $I't(t)$ truncated to 7 terms, superimposed on a graph of $I't(t)$ plotted as a dotted line.

```
* fct s(t)=sum(i,1,n, d(i,2)*cos(2*pi*d(i,1)*t + d(i,3)) )
* n=7
* q=points(s,0:25!120)
* delete w,w1
* draw m col 1:2 lt dotted
* draw q
* view
```