Computing a Physical Spline Curve

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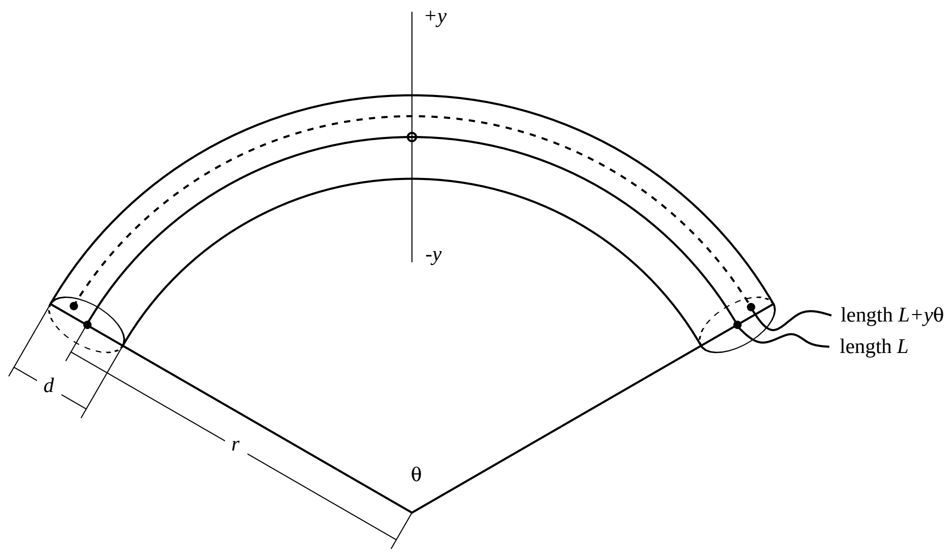
A flexible springy wire of length h lying in the xy-plane pinned at the end-points (a_1, a_2) and (b_1, b_2) with or without specified directions at these end-points is physically determined to have one of several stable shapes, each corresponding to a locally-minimal energy value. We may seek these minimal energy shapes; such a curve will be called a (planar) *physical spline* curve. Although a cubic spline segment may be a good approximation to a physical spline, this is often not the case, and moreover the length constraint applied to a cubic spline segment is difficult to honor. It is important in various architectural and engineering applications (*e.g.* building, ship, airplane and automobile design) to be able to compute the exact physical spline curve of a given length that satisfies given boundary conditions.

Our first step is to compute the energy in a thin circular-cross-section physical spline bent in the shape of an arc-length parametrized space curve x of length h. We will model the spline by a sequence of n segments bent in circular arcs and adjoined end-to-end to approximate the curve x. Each segment is of length $L = h/n$. A segment is modeled by a bundle of elastic length-L fibers arranged to form a cylinder of length L. Each fiber behaves as a spring that is compressed or extended in length with a force proportional to the change in its length according to Hooke's law.

As each cylindrical segment of fibers is bent into a circular arc, the length of the central fiber is unchanged in length, the fibers closer to the inside of the arc are compressed, and the fibers on the outside of the arc are extended. The potential energy stored in all these compressed and extended fibers is the energy of the segment, and the sum of the energies of the n segments approximates the energy in the spline bent to follow the curve x. When we consider the limiting case where $n \to \infty$, we obtain the desired energy of the physical spline; this will also be defined to be the energy of the curve x.

The potential energy e in a length-L spring fiber compressed or extended to the length $L+\Delta$ is the work done to change the length from the 0-energy length L to the length $L+\Delta$. This work is the quantity $\int_L^{L+\Delta} F(x)\,dx$ where $F(x)$ is the force needed to compress or extend the spring to length x. By Hooke's law $F(x) \approx \varepsilon(x-L)/L$ when x is not too different from L, where ε is the modulus of elasticity of the spring material. Thus the energy e is $\frac{\varepsilon}{2L}\Delta^2$.

Now let us consider the length-L cylindrical segment of fibers bent in a circular arc of a circle of radius r as shown below. The origin is shown placed at the center of the cylindrical segment.



The central fiber of length L is shown. A coordinate axis is established by which we may measure the distance y above or below the central fiber. A representative non-central fiber of length $L+\theta y$ is also shown.

The values r, L, and θ are related by $r\theta = L$. The curvature of the central fiber is $1/r$. In general, the length of a fiber offset vertically by the distance y is $(r+y)\theta$. The energy of the fiber offset vertically by the distance y is thus $\varepsilon(y\theta)^2/(2L)$.

Let A denote the circular disk of diameter d centered at $(0,0)$ in the xy-plane. The total energy stored in the entire cylindrical segment is now obtained as the integral

$$\int_A \varepsilon(y\theta)^2/(2L)\,dxdy = \frac{\varepsilon}{2L}\theta^2\int_A y^2\,dxdy.$$

But $\int_A y^2\,dxdy$ is a constant which we call I; in fact I is the so-called moment of inertia of the disk A about the x-axis. Thus the energy in our bent segment is $\frac{\varepsilon I}{2L}\theta^2 = \frac{\varepsilon I}{2}\cdot\frac{L}{r^2}$. Since $1/r$ is the curvature of the central fiber, we may write the energy of the segment as $\frac{\varepsilon I}{2}\cdot Lk^2$, where $k = 1/r$ denotes the curvature of the segment.

Now define the arc-length parameter value $s_i := (i-1)L+L/2$. Let $K(s)$ denote the curvature $|x''(s)|$ of our given arc-length parametrized curve x at s.

Place n length-L cylindrical segments along the curve x with the ith segment centered at $x(s_i)$. The ith segment is taken to be bent in a circular arc whose curvature matches the curvature value $K(s_i)$. When n is large, the circular end-faces of the adjacent segments will approximately join.

The total potential energy E in all n segments is then

$$\sum_{1\le i\le n}\frac{\varepsilon I}{2}(K(s_i))^2 L.$$

Now, if we let $n\to\infty$, then $L\to 0$ such that nL remains equal to the curve length h, and the summation expression for E is seen to be a Riemann sum which converges to the integral $\frac{\varepsilon I}{2}\int_0^h K(s)^2\,ds$.

Thus the energy in a diameter-d circular cross-section physical spline bent to follow the curve x is $\frac{\varepsilon I}{2}\int_0^h K(s)^2\,ds$. We see that a physical spline with an infinitesimal cross-section diameter has an infinitesimal energy; however we shall depart from physical reality and *assign* the value $\int_0^h K(s)^2\,ds$ to be the mathematical energy of the length h curve x.

Note that a ruled-surface composed of a rectangle of stiffly-flexible material which admits a family of parallel rule-lines bent according to uniform constant boundary conditions along two opposing edges formed by rule-lines assumes the same shape in any individual cross-section curve and that shape is the shape of a planar physical spline curve determined by the distance between the two opposing edges and the boundary conditions at those edges.

Now we may consider the problem of computing the shape of the minimum-energy physical spline of length h which connects the two given points $a = (a_1, a_2)$ and $b = (b_1, b_2)$. We may optionally impose tangent vector direction constraints at one or both of these points. The desired curve x can be defined as that curve which has minimal energy, subject to the required constraints. Thus, x is to be chosen so that the functional

$$\mathcal{E}(x) := \int_0^h |x''(s)|^2 \, ds$$

is minimal, subject to the constraints: $x(0) = a$, $x(h) = b$, and $\int_0^v |x'(s)| \, ds = v$ for $0 \le v \le h$, or equivalently, $|x'(s)| = 1$ for $0 \le s \le h$. This latter set of constraints forces x to be an arc-length parameterized curve of overall length h for which the curvature $K(s) = |x''(s)|$. If tangent vector constraints of the form $x'(0) = y/|y|$ and/or $x'(h) = z/|z|$ are to be imposed, then they must be added to the primary set of constraints just given.

Each curve x corresponding to a local minimum of $\mathcal{E}$ is a stable shape for the length-h physical spline connecting the points a and b. Every initially-selected curve in a small-enough neighborhood of x as defined by the functional $\mathcal{E}$ will relax to the shape x for which $\mathcal{E}$ is locally-minimal with an accompanying loss of energy.

There are several approaches to computing the planar physical spline curve x that minimizes $\mathcal{E}(x)$. One approach, due to Mehlum, and studied by Mehlum, Kallay, and Jou ([Meh74], [Kal86], [Jou95]), is to define x by its curvature function, and then to obtain the Euler differential equations that define the curve x that minimizes the associated energy via the calculus of variations. These differential equations have associated boundary conditions as well as several unknown parameters which must be computed via minimization. The reason we restrict ourselves here to the planar case is that this approach is more cumbersome for a general space curve (however see [Kno98] for the general space-curve result.)

Now, to compute the functions x_1 and x_2 that define the minimal energy length-h physical spline connecting a and b, we may proceed as follows. Let $\theta(s)$ denote the direction-angle of the tangent-vector of the arc-length-parameterized planar curve x at s. Then the curvature of x, K, is $|\theta'|$, so the corresponding energy is $E = \int_0^h (\theta'(s))^2 \, ds$.

Note $x_1'(s) = \cos(\theta(s))$ and $x_2'(s) = \sin(\theta(s))$. Thus

$$\begin{aligned} \int_0^h x_1'(s) ds &= x_1(h) - x_1(0) = \int_0^h \cos(\theta(s)) ds, \text{ and} \\ \int_0^h x_2'(s) ds &= x_2(h) - x_2(0) = \int_0^h \sin(\theta(s)) ds. \end{aligned}$$

We wish to find the function θ that minimizes E subject to the directional constraints: $\theta(0) = c$ and $\theta(h) = d$ for given vectors c and d, together with the constraints: $\int_0^h \cos(\theta(s))ds = b_1 - a_1$ and $\int_0^h \sin(\theta(s))ds = b_2 - a_2$. Given the function θ, we can then compute the curve x as the solution to the differential equations $x_1'(s) = \cos(\theta(s))$ and $x_2'(s) = \sin(\theta(s))$, with $x_1(0) = a_1$ and $x_2(0) = a_2$.

Kallay [Kal86] and Jou [Jou95] have computed the Euler equation that defines θ to be

$$\theta'' + m_1 \sin(\theta) - m_2 \cos(\theta) = 0,$$

with the boundary conditions $\theta(0) = c$ and $\theta(h) = d$, where m_1 and m_2 are unknown Lagrange multipliers to be determined.

This differential equation arises in a specialized form in describing the shape of a bent beam in structural engineering. There we have a beam of length h with one end fixed at the origin with slope 0, and with a load or force f applied at the other end of the beam. Then the angle of the tangent along the beam, θ, satisfies the differential equation $\theta'' = -\frac{f_2}{\epsilon I}\cos(\theta) - \frac{f_1}{\epsilon I}\sin(\theta)$ with $\theta(0) = \theta'(0) = 0$. The value ϵ is the modulus of elasticity of the beam material and I is the moment of inertia of the cross-section of the beam about its horizontal axis. Solutions of this differential equation have been obtained in terms of incomplete elliptic integrals [Jou95].

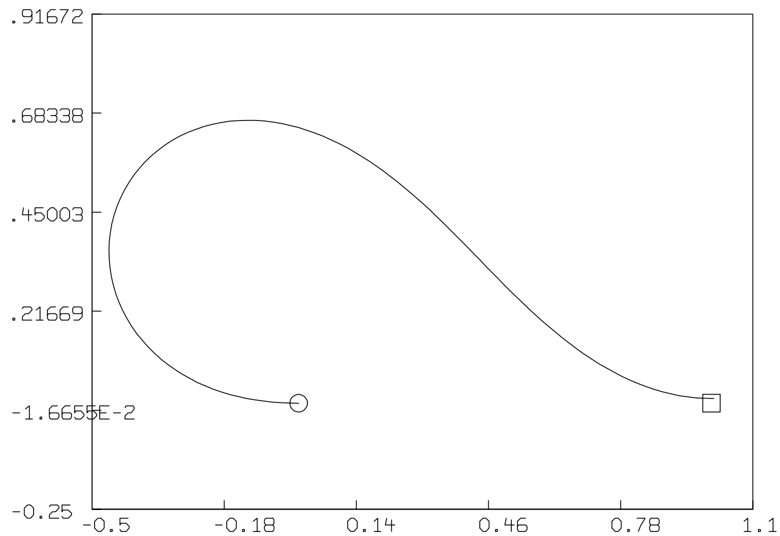
The MLAB mathematical and statistical modeling software is well-suited to studying differential equation models, especially when parameter-estimation is required (see `www.civilized.com`). Below we show the MLAB commands for computing and drawing a planar physical spline curve that connects two given points with given tangent vector directions at those points. This is done by specifying the appropriate differential equations and using the MLAB curve-fitting facility to compute the parameter-values which fit the model functions defined by these differential equations to the given points and directions.

We are given the vectors a and b, and the scalars c and d, and we are to compute the values $\theta(0)$, $\theta'(0)$, $x_1(0)$, $x_2(0)$, m_1, and m_2 which yield the desired curve x. Of course, $x_1(0) = a_1$, $x_2(0) = a_2$ and $\theta(0) = c$. The remaining values $\theta'(0)$, m_1 and m_2 can be determined by curve-fitting x_1 to (h, b_1) and x_2 to (h, b_2) and θ to (h, d) in MLAB. Jou [Jou95] gives a survey of planar physical splines and groups them corresponding to the values of m_1 and m_2; this provides a strategy for obtaining initial guesses for m_1 and m_2.

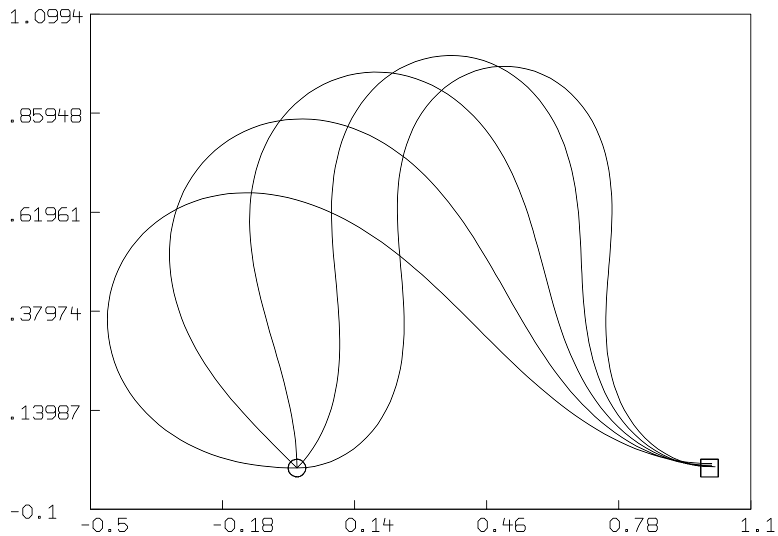
```
* fct t''s(s)=m1*x1's(s) + m2*x2's(s)
* fct x1's(s) = cos(t)
* fct x2's(s) = sin(t)
```

```
* init x1(0)=a1
* init x2(0)=a2
* init t(0)=c
* init t's(0)=v

* a1=0; a2=0
* b1=1; b2=0
* c=pi; d=0
* h=2.5

* tolsos=.0001
* errfac=.00001

* m1=3.7; m2=-1.04; v=-1.25

* fit(m1,m2,v), t to h&'d, x1 to h&'b1, x2 to h&'b2
final parameter values
      value               error                dependency    parameter
   3.6905090629      4.065013222e-05          0.8832055969   M1
  -1.0704404889      7.016612564e-05          0.9513330028   M2
  -1.2518649953      6.139471472e-05          0.8554696171   V
6 iterations
CONVERGED
best weighted sum of squares = 2.717760e-10
weighted root mean square error = 1.648563e-05
weighted deviation fraction = 6.297892e-06
R squared = 1.000000e+00

* draw a1&'a2 pt circle
* draw b1&'b2 pt square
* z=integrate(x1's,x2's,0:h!120)
* draw z col (2,4)
* view
```

Here is the computed length 2.5 physical spline that goes from $(0, 0)$ with the exit-angle π to the point $(1, 0)$ with the entry-angle 0.



Below we show the five length 2.5 physical splines that connect $(0,0)$ to the point $(1,0)$ with the entry-angle 0, where the exit-angle ranges over $\{\pi, \frac{3\pi}{4}, \frac{\pi}{2}, \frac{\pi}{4}, 0\}$.



References

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