

Analysis of *the Rule of 70* with MLAB

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Abstract

The Rule of 70 states that the time required for a fixed income investment to double is approximately $70/R$, where R is the fixed interest rate per compounding period. The reliability of *the Rule of 70* is analyzed. The rule is exact when $R \approx 1.9838$; the rule overestimates the doubling time when $R < 1.9838$; and the rule underestimates the doubling time when $R > 1.9838$. The value 70 is found to be the nearest multiple of 10 for the parameter a in the function a/R that best approximates, in the least squares sense, the exact doubling time. *The Rule of 70* is within 1 percent of the exact doubling time for $0 < R < 4.02353076$, within 5 percent of the exact doubling time for $0 < R < 12.3144692$, and within 10 percent of the exact doubling time for $0 < R < 22.9657819$.

The value of a fixed interest rate investment with principal P , and interest rate R , after n compounding periods, is given by:

$$\left(1 + \frac{R}{100}\right)^n P$$

The Rule of 70 is a rule of thumb that provides a simple and convenient method for estimating the time required for an investment paying a fixed rate of return to double in value. The exact time required for an investment to double in value is found by solving the following equation for n :

$$2P = \left(1 + \frac{R}{100}\right)^n P$$

The solution is: $n = \frac{\log(2)}{\log(1 + \frac{R}{100})}$. *The Rule of 70* states that if an investment returns an amount equal to the percentage R of the amount currently

invested in each period of time over the life of the investment, then the value of the investment, comprised of principal and compounded interest, will double the initial value after approximately $(70/R)$ periods of time, i.e.

$$n = \frac{\log(2)}{\log(1+\frac{R}{100})} \approx \frac{70}{R}$$

As an example, consider an investment returning 5 percent each year. According to *the Rule of 70*, the principal and interest will double the initial principal value after $70/5 = 14$ years. A more accurate estimate for the doubling time for an investment returning 5 percent each year is 14.2066991 years; in this case *the Rule of 70* provides a value that is on the low side but correct to 1 part in 100.

However *the Rule of 70* is not always so accurate. As another example, consider an investment returning 100 percent each year. According to *the Rule of 70*, the principal and interest will double the initial principle after $70/100 = 0.7$ years. The actual doubling time for an investment returning 100 percent each year is one year. So here *the Rule of 70* is significantly lower than the exact doubling period.

The MLAB mathematical modeling computer program is a scriptable tool that allows the user to explore algebraic, differential equation, and implicit-function models, to name just a few applications. This paper demonstrates its use in determining the range of validity of *the Rule of 70*. The MLAB program is available for Windows, Linux, and PPC-Macintosh systems. The MLAB program presents the user with a window in which commands are typed and numerical responses from MLAB are generated. MLAB also features a variety of graphics commands that allow the user to create another window to display graphical output. Scripts of MLAB commands called do-files, are runnable with the command `D0 F`.

The following text shows the MLAB commands for computing the doubling time of an investment according to *the Rule of 70* and compares the result to the exact formula. (Note, comments in the lists of MLAB commands that follow that are delimited by `/*` and `*/` are ignored by MLAB.)

```
/* Define a function for the exact doubling time for a given
   interest rate.*/
fct d(r) = log(2)/log(1+r/100)
```

```

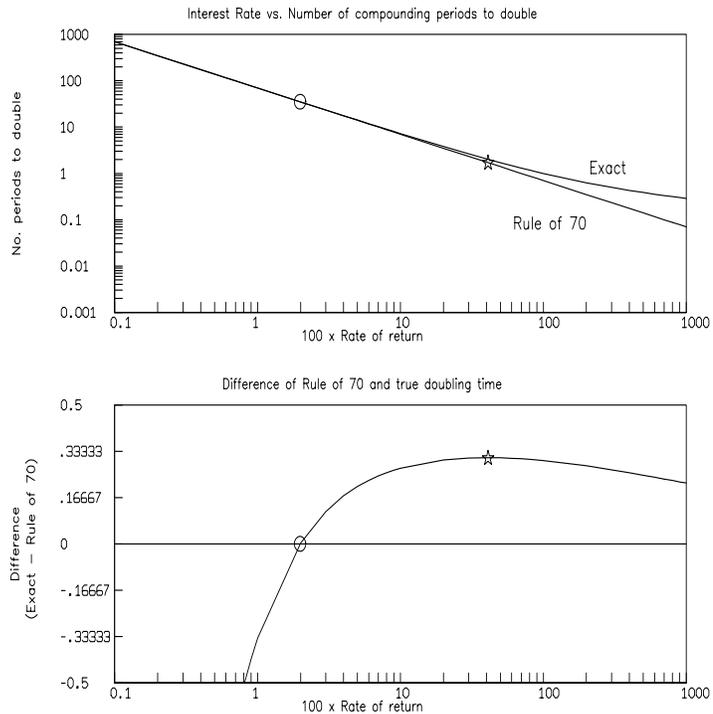
/* Define a function for the doubling time according to the Rule of 70.*/
fct e(r) = 70/r

/* Define a function for the difference between the Rule of 70 and
   exact doubling times.*/
fct f(r) = e(r)-d(r)

/* Draw graphs of the functions d(r) and e(r) in window w1, and
   the difference function d(r)-e(r) in window w2, over four ranges
   of interest rates.*/
r = (.1:.9:.1)&(1:9:1)&(10:90:10)&(100:900:100)&1000
draw points(d,r) in w1
draw points(e,r) in w1
draw points(f,r) in w2

```

With suitable commands (not shown) to make the MLAB graphical window w1 have log-log axes and the MLAB graphical window w2 have log-linear axes, the following picture is obtained:



Two features in these graphs are apparent and highlighted by a circle and a star. First, the lower graph shows there is an interest rate at which *the Rule of 70* doubling time equals the exact doubling time. This point is where the difference curve intersects the line $y = 0$ and is marked by a circle. The numerical value of the interest rate at the point of intersection can be determined in MLAB as follows:

```
/* Define a function for the difference between the Rule of 70 and
   exact doubling expressions.*/
fct z(t) = ROOT(r,0.001,100,e(r)-d(r))
type z(0)
```

The ROOT command searches the interval of r -values from 0.001 to 100 for a value where $e(r)-d(r)$ is zero. MLAB responds to these commands with:

```
= 1.98380058
```

So the doubling time as determined by *the Rule of 70* is exact if the interest rate is 1.98380058 percent. It is apparent from the lower graph that at interest rates less than 1.98380058 percent, *the Rule of 70* overestimates the exact doubling time and at interest rates larger than 1.98380058 *the Rule of 70* underestimates the exact doubling time.

The lower graph also shows that although there is no lower limit to the difference, there is an upper limit to the difference. The upper limit to the difference, marked on the graphs with a star, can be found by obtaining the interest rate at which the derivative of the difference of *the Rule of 70* doubling time from the exact doubling time—with respect to interest rate, is zero. The interest rate at which the derivative is zero can be computed with the following MLAB commands:

```
/* Compute the interest rate at which the error is a maximum.*/
fct y(t) = ROOT(r,0.001,100,e'r(r)-d'r(r))
type y(0)
```

Note that in the MLAB language, the derivative of a function, $f(t)$, with respect to its argument, t , is expressed as $f't(t)$. MLAB responds to these commands with:

```
= 41.0242057
```

This response indicates that the maximum underestimate of the doubling time according to *the Rule of 70* occurs with the interest rate equal to 41.0242057 percent.

MLAB also provides a non-linear curve fitting command which allows one to determine the value of the proportionality constant, a , in the expression $e(r) = a/r$ that has the least sum of squares error. The following series of commands determine the best value:

```
/* Define a function to fit the doubling function that is inversely
   proportional to the rate of return.*/
fct g(r) = a/r

/* Provide an initial guess for the value of the parameter.*/
a = 1
```

```

/* Generate a matrix with exact data points over a wide range of
   interest rates.*/
mr = (.1:.9:.1)&(1:9:1)&(10:90:10)
mr = points(d,mr)

/* Reduce the sum of squares by adjusting parameter a.*/
fit (a), g to mr

```

In response to these commands, MLAB responds:

```

final parameter values
      value          error          dependency  parameter
      69.38459718    0.02400283349              0      A
2 iterations
CONVERGED
best weighted sum of squares = 2.329796e+00
weighted root mean square error = 2.993452e-01
weighted deviation fraction = 6.583745e-04
R squared = 9.999959e-01

```

This response indicates that the value of the proportionality factor that minimizes the sum of squares error between the data in the matrix `mr` and the model function `g` is 69.38459718. The nearest multiple of ten to this value is the value used in *the Rule of 70*.

Finally, we can use MLAB to find the maximum interest rate at which *the Rule of 70* is within 10, 5, and 1 percent of the exact doubling time as follows:

```

/* Compute the interest rate at which Rule of 70 is within %10,
   %5, and %1 of the exact doubling time.*/
fct w(t) = ROOT(r,0.001,100,t-(d(r)-e(r))/e(r))
type w(.1), w(.05), w(.01)

```

MLAB responds to these commands as follows:

```

= 22.9657819
= 12.3144692
= 4.02353076

```

These numbers mean that *the Rule of 70* is within:

1 percent of the actual doubling time for interest rates in the interval $[0, 4.02353076]$;
5 percent of the actual doubling time for interest rates in the interval $[0, 12.3144692]$; and
10 percent of the actual doubling time for interest rates in the interval $[0, 22.9657819]$.

Other applications of MLAB in the area of business, accounting, and finance include analyses of:

- the Dow Jones industrial average from 1990 to the present,
- the election to collect Social Security at age 65 or 70.5,
- amortization schedule computations, and
- inventory replacement schedules based on survival functions.

For more information about these applications and MLAB, please contact Civilized Software at:

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